Phase-plane Analysis of Ordinary Differential Equations

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Seminar in Engineering Analysis

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California State University Northridge

Outline

- Midterm exam two weeks from tonight covering ODEs and Laplace transforms
- Review last class
- Introduction to phase-plane analysis
- Look at two simultaneous ODEs dy_1/dt and dy_2/dt plotted as y_1 vs. y_2
- Look at different "critical points" for different systems of equations

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Review Laplace Transforms

- Use transform tables to transform terms in differential equation for y(t) into an algebraic equation for Y(s)
  - Derivative transforms give initial conditions on y(t) and its derivatives
- Manipulate Y(s) equation to sum of individual terms, "Y(s) subterms", that you can find in transform tables
  - Manipulation may require use of method of partial fractions

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Review Partial Fractions

- Method to convert fraction with several factors in denominator into sum of individual factors (in denominator)
- Example is F(s) = 1/(s+a)(s+b)
- Write 1/(s+a)(s+b) = A/(s+a) + B/(s+b)
- Multiply by (s+a)(s+b) and equate coefficients of like powers of s
  - 1 = A(s + b) + B(s + a)
  - A + B = 0 for s^1 terms and 1 = bA + aB for s^0 terms

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Review Partial Fractions II

  - A + B = 0 for s^1 terms and 1 = bA + aB for s^0 terms
  - Solving for A and B gives A = -B = 1/(b – a)
- Result: 1/(s+a)(s+b) = 1/[(b – a)(s + a)] – 1/[(b – a)(s + b)]
  - So f(t) = $[e^{-at} - e^{-bt}]/(b - a)$
- This actually matches a table entry
- Follow same basic process for more complex fractions
- Special rules for repeated factors and complex factors

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Review Partial Fraction Rules

- Repeated fractions for repeated factors

$$\frac{1}{\cdots(s+a)^n\cdots} = \cdots + \frac{A_n}{(s+a)^n} + \frac{A_{n-1}}{(s+a)^{n-1}} + \cdots + \frac{A_2}{(s+a)^2} + \frac{A_1}{s+a} + \cdots$$

- Complex factors (s + α – iβ)(s + α + iβ)

$$\frac{1}{\cdots(s+\alpha-i\beta)(s+\alpha+i\beta)\cdots} = \cdots + \frac{As+B}{(s+\alpha)^2+\beta^2} + \cdots$$

- Pure imaginary factor

$$\cdots + \frac{1}{s^2+\beta^2} + \cdots = \frac{1}{\cdots(s-i\beta)(s+i\beta)\cdots} = \cdots + \frac{As+B}{s^2+\beta^2} + \cdots$$

- Real squared factors

$$\cdots + \frac{1}{s^2-\beta} + \cdots = \cdots + \frac{A}{\left(s+\sqrt{\beta}\right)}\frac{B}{\left(s-\sqrt{\beta}\right)} + \cdots$$

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Review Systems of ODEs

- Apply Laplace transforms to systems of equations by transforming all ODEs
  - Transform ODE terms like y_k to $Y_k(s)$, dy_k/dt to $sY_k(s) - y_k(0)$, etc.
- Transform all ODEs in system then use Gaussian elimination to get an equation for only one $Y_k(s)$
- Get inverse transform from $Y_k(s)$ to $y_k(t)$
- Repeat for all ODEs

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Review Group Exercise

- Solve $y'' - 9y = e^{-t}$ with $y(0) = 0$ and $y'(0) = 2$ by Laplace transforms
- Transform differential equation:

$s^2Y(s) - sy(0) - y'(0) - 9Y(s) = 1/(s+1)$

- Substitute initial conditions and solve result for Y(s)

$s^2Y(s) - 0 - 2 - 9Y(s) = 1/(s+1)$

$(s^2 - 9)Y(s) = 2 + 1/(s+1)$

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Review Group Exercise II

$(s^2 - 9)Y(s) = 2 + 1/(s+1)$

$$Y(s) = \frac{2}{s^2-9} + \frac{1}{(s^2-9)(s+1)}$$

- Use partial fractions for last term

$$\frac{1}{(s^2-9)(s+1)} = \frac{A}{(s-3)} + \frac{B}{(s+3)} + \frac{C}{(s+1)}$$

$$1 = A(s+1)(s+3) + B(s+1)(s-3) + C(s^2-9)$$

- Set sums of like powers to zero

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Review Group Exercise III

$$1 = A(s+1)(s+3) + B(s+1)(s-3) + C(s^2-9)$$

s^2 ***terms:*** $0 = A + B + C$

s^1 ***terms:*** $0 = 4A - 2B$

s^0 ***terms:*** $1 = 3A - 3B - 9C$

- s^1 equation gives B = 2A
- Substituting B = 2A into s^2 equation gives A + 2A + C = 0 or C = –3A
- Substitute B = 2A and C = –3A into s^0 equation to get 1 = 3A – 3(2A) – 9(–3A)

A = 1/24

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Review Group Exercise IV

- From A = 1/24 and B = 2A: B = 2/24
- From A = 1/24 and C = -3A: C = -3/24

$$Y(s) = \frac{2}{s^2-9} + \frac{A}{(s-3)} + \frac{B}{(s+3)} + \frac{C}{(s+1)}$$

$$Y(s) = \frac{2}{s^2-9} + \frac{1}{24}\left[\frac{1}{(s-3)} + \frac{2}{(s+3)} - \frac{3}{(s+1)}\right]$$

- From transform table

$$\boxed{y(t) = \frac{2}{3}\sinh(3t) + \frac{1}{24}\left[e^{3t} + 2e^{-3t} - 3e^{-t}\right]}$$

$$\sinh(x) = [e^x + e^{-x}]/2$$

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Basic Phase Plane Equations

- Look at solution of system of two first-order autonomous (no t dependence) homogenous equations
- Can be single second order equation written as two first order equations

$$\frac{d^2y}{dt^2} + \frac{c}{m}\frac{dy}{dt} + \frac{k}{m}y = 0 \qquad v = \frac{dy}{dt}$$

$$\frac{dv}{dt} + \frac{c}{m}v + \frac{k}{m}y = 0$$

$$\frac{dy}{dt} - v = 0$$

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Phase Plane Analysis

- Look at solutions of systems of equations, here use two equations as an example
- Find certain points, called critical points, that have particular behavior depending on the eigenvalues of the ODE's
- This leads to a discussion of stability; will a solution tend to zero or increase without bound?

What is a Stable Solution?

- Plot trajectories (plot one dependent variable against the other)
  - For 2 ODE's dy/dt and dv/dt, plot y vs. v
- If at one time the trajectory is within a distance ε of a point P_0 and for all future times it remains within a distance δ of P_0, the solution is stable
- The solution is unstable if it is not stable
- Want to find criteria for stable solutions

Undamped Vibrations Example

- Equation: $d^2x/dt^2 + \omega^2 x = 0$ ($\omega^2 = k/m$)
  - Solution: $x = (v_0/\omega)\sin \omega t + x_0 \cos \omega t$
  - As system of equations $dx/dt = v$ and $dv/dt = d^2x/dt^2 = -\omega^2 x$
  - Define $y_1 = x$ and $y_2 = v$ to get system of equations as $dy_1/dt = y_2$ and $dy_2/dt = -\omega^2 y_1$

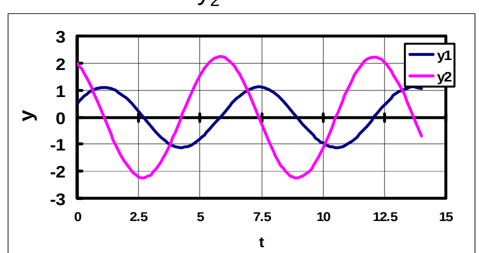
$$y_1' = a_{11}y_1 + a_{12}y_2 = y_2$$

$$y_2' = a_{21}y_1 + a_{22}y_2 = -\omega^2 y_1$$

$$a_{11} = a_{22} = 0 \qquad a_{12} = 1 \qquad a_{21} = -\omega^2$$

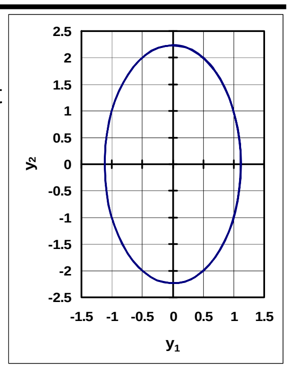
Phase Plane Introduction

- Usual plot shows solutions for $x = y_1$ and $v = dx/dt = y_2$ as a function of time

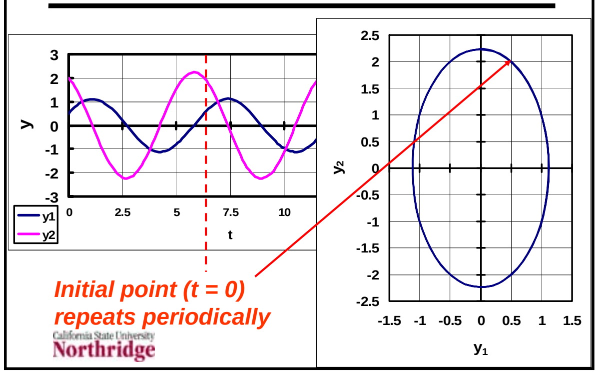


Phase Plane Introduction II

- Phase plane plot shows y_2 as a function of y_1 with t as a parameter
- Same as previous plot y_1 is displacement and y_2 is velocity
  - Time differs along plot



Phase Plane Introduction III



General Form

- Write as matrix equation d**y**/dt = **Ay**
- General form for two equations and solutions in terms of **Ax** = λ**x** eigenvalues and eigenvectors is

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y} \quad \Rightarrow \quad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 \\ y_2' &= a_{21}y_1 + a_{22}y_2 \end{aligned}$$

$$\mathbf{y} = C_1\mathbf{x}_{(1)}e^{-\lambda_1 t} + C_2\mathbf{x}_{(2)}e^{-\lambda_2 t} \qquad \begin{aligned} y_1 &= C_1 x_{1(1)}e^{-\lambda_1 t} + C_2 x_{1(2)}e^{-\lambda_2 t} \\ y_2 &= C_1 x_{2(1)}e^{-\lambda_1 t} + C_2 x_{2(2)}e^{-\lambda_2 t} \end{aligned}$$

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Stable Solution Criteria

- Look at the system of two equations d**y**/dt = **Ay**
  - Autonomous systems (no t dependence)
- We will show that stability depends on the trace of **A** = $a_{11} + a_{22} = \lambda_1 + \lambda_2$, and the determinant $a_{11}a_{22} - a_{12}a_{21}$ and the discriminant, Δ = (trace **A**)^2 – 4det **A**
- Review eigenvalues for 2 x 2 matrix from September 12 lecture

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Two-by-two Matrix Eigenvalues

- Quadratic equation with two roots for eigenvalues

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12} =$$

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{21}a_{12} = 0$$

- Eigenvalue solutions

Trace **A**

$$\lambda = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{21}a_{12})}}{2}$$

Det **A**

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Return to Previous Example

- For undamped vibrations we had

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 = y_2 \\ y_2' &= a_{21}y_1 + a_{22}y_2 = -\omega^2 y_1 \end{aligned}$$

$$a_{11} = a_{22} = 0 \qquad a_{12} = 1 \qquad a_{21} = -\omega^2$$

- This gives trajectory slope as

$$\frac{dy_2}{dy_1} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2} = \frac{-\omega^2 y_1 + 0y_2}{0y_1 + (1)y_2} = -\frac{\omega^2 y_1}{y_2}$$

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Continue Undamped Vibration

- Have separable solution giving equation for ellipse

$$\frac{dy_2}{dy_1} = -\frac{\omega^2 y_1}{y_2} \quad \Rightarrow \quad y_2 dy_2 = -\omega^2 y_1 dy_1$$

$$\frac{y_2^2}{2} = -\omega^2\frac{y_1^2}{2} + C \quad \Rightarrow \quad y_2^2 + \omega^2 y_1^2 = y_{2,0}^2 + \omega^2 y_{1,0}^2$$

$$\frac{y_2^2}{\omega^2 y_{1,0}^2 + y_{2,0}^2} + \frac{y_1^2}{y_{1,0}^2 + \frac{y_{2,0}^2}{\omega^2}} = \frac{v^2}{\omega^2 x_0^2 + v_0^2} + \frac{x^2}{x_0^2 + \frac{v_0^2}{\omega^2}} = 1$$

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What Happens if $y_1 = y_2 = 0$?

- Autonomous system of two equations

$$\frac{dy_2}{dy_1} = \frac{dy_2/dt}{dy_1/dt} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2}$$

- Trajectory slope, dy_2/dy_1, depends on values of **A** and may be indeterminate at $y_1 = y_2 = 0$
- $y_1 = y_2 = 0$ is called a **critical point**
  - For multidimensional systems **y** = **0**

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Types of Critical Points

- Critical points are points on a y_1-y_2 plot that are classified depending on the trajectory shapes at or near these points
- Centers
- Improper Nodes
- Proper Nodes
- Saddle Points
- Spiral Points

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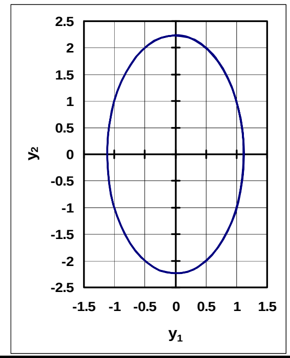
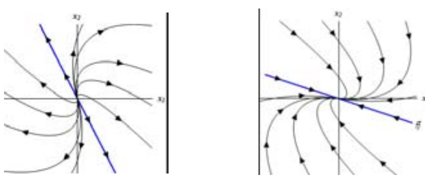
Our First Critical Point

- The undamped vibrations solution is an ellipse that does not go through $y_1 = y_2 = 0$
- This type of critical point is called a center



Improper Node



- All trajectories, except two of them have the same limiting direction of the tangents
- The two exceptions will have a different direction

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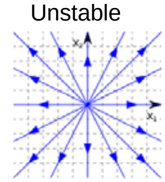
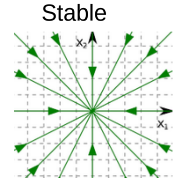
Proper Node

Unstable



Stable



$$\boldsymbol{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{x} \qquad \boldsymbol{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t$$

https://upload.wikimedia.org/wikipedia/commons/4/49/Phase_Portrait_Unstable_Proper_Node.svg

https://upload.wikimedia.org/wikipedia/commons/4/4a/Phase_Portrait_Stable_Proper_Node.svg

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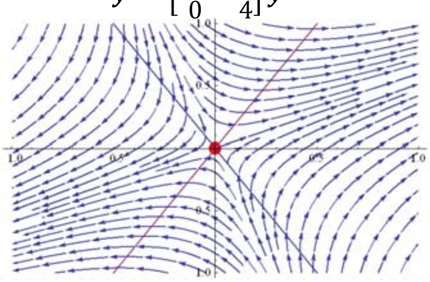
Unstable Saddle Point

$$\boldsymbol{y}' = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \boldsymbol{y}$$



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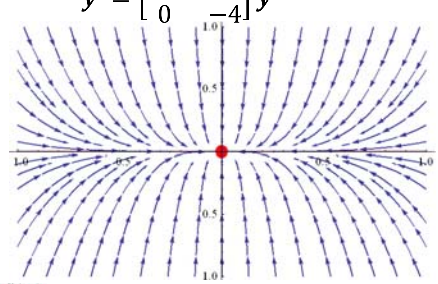
Stable Saddle Point

$$\boldsymbol{y}' = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \boldsymbol{y}$$



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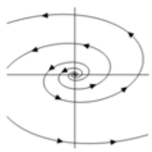
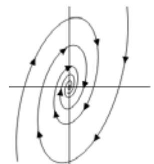
Spiral Points



Asymptotically Unstable



Asymptotically Stable

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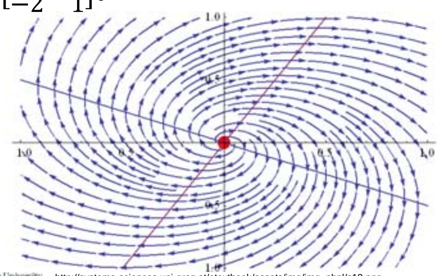
Unstable Spiral Source

$$\boldsymbol{y}' = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \boldsymbol{y}$$



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